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MAM1020S **Tutorial 7** **August 2017**

1. Find the derivative of each function. Give also the domain of definition of the function and the derivative.

   SOMETIMES IT IS GOOD TO WORK ON f A BIT IN ORDER TO COMPUTE THE DERIVATIVE. BE AWARE THAT WHEN CHANGING THE FORM OF A FUNCTION YOU OBTAIN A DIFFERENT FUNCTION PERHAPS IN THE SENSE THAT THE DOMAIN OF DEFINITION MIGHT CHANGE.

   —¿ IN BOLD YOU FIND WHAT I THINK IN MY HEAD SOMETIMES

   1. $f(x) = 10x^{2.1} + 365$
      Solutions: f is a polynomial function thus its domain of definition of f and the domain of definition of its derivative is $\mathbb{R}$.
      Using the power rule we get:
      $$f'(x) = 21x^{1.1}$$
   2. $f(r) = \frac{4}{3}\pi r^3$
      Solutions: f is a polynomial function thus its domain of definition of f and the domain of definition of its derivative is $\mathbb{R}$.
      Using the power rule we get:
      $$f'(r) = 4\pi r^2$$
   3. $f(x) = -x^3 + 2x^2 - 6$
      Solutions: f is a polynomial function thus its domain of definition of f and the domain of definition of its derivative is $\mathbb{R}$.
      Using the power rule we get:
      $$f'(x) = -3x^2 + 4x$$
   4. $f(t) = \frac{4}{t^4} - \frac{3}{t^3} + \frac{2}{t}$
      Solutions: f is a sum of rational functions thus its domain of definition of f is the same as the domain of definition of its derivative it is the set of real values such that the denominator of the rational function is non zero, that is $\mathbb{R}^*$.
      Using the power rule we get:
      $$f'(t) = -\frac{16}{t^5} + \frac{9}{t^4} - \frac{2}{t^2}$$
   5. $f(x) = \sqrt{x^3} - 3x^{-2} + 2x^{-\frac{12}{13}}$
      Solutions: f is a sum of a square root function with rational functions. Since the polynomial is defined over $\mathbb{R}$ and the square root $\sqrt{x^3}$ for any $x \in \mathbb{R}$ such that $x^3 \geq 0$ which is equivalent to $x \geq 0$ and the rational function are defined when their denominator is non zero that is $x \neq 0$, then the domain of definition of f is $(0, \infty)$.
      Note that $f(x) = x^{\frac{3}{2}} - 3x^{-2} + 2x^{-\frac{12}{13}}$
      $$f'(x) = 3/2\sqrt{x} + 6x^{-3} - \frac{24}{13}x^{-\frac{25}{13}}$$
      Thus as before, the domain of definition of the derivative is $(0, \infty)$.
   6. $f(x) = \frac{3x-5x^3}{\sqrt[3]{x}}$
      Solution
      f is the quotient of a polynomial by a square root function, thus defined when the square root function is non zero and what is behind the square root is positive, that is that the domain of definition of f is $(0, \infty)$.
      Note that
      $$f(x) = 2x^{2/3} - 5x^{8/3},$$
      thus using the power rule we get
      $$f'(x) = 4/3x^{-1/3} - \frac{40}{3}x^{8/3}$$

7. $f(z) = \sqrt{5}z + \sqrt{11z}$
Solution f is a sum of a polynomial function with a square root function thus defined when what is behind the square root is positive, that is $z \geq 0$.
Note that
$$f(z) = \sqrt{5}z + \sqrt{11z} = \sqrt{5}z + \sqrt{11}\sqrt{z}$$
Thus, using the power rule we get:
$$f'(z) = \sqrt{5} + \sqrt{11}/2z^{-1/2}$$
The domain of definition of f' is now excluding 0 because $x \mapsto \sqrt{11}/2z^{-1/2}$ is now a rational function, thus the domain of f' is $(0, \infty)$.

8. $f(x) = \frac{(3x+1)x}{\sqrt{x}}$
Solution f is a quotient of a polynomial and a square root function thus defined for the real x values such that $x > 0$.
Note that $f(x) = (3x+1)\sqrt{x} = 3x^{3/2} + \sqrt{x}$ thus using the power rule we get:
$$f'(x) = 9/2x^{1/2} + 1/2x^{-1/2}$$
And the domain of definition of f' is the same as the domain of f, for the same reason as explained before.

9. $f(x) = 3tan(x) - sec(x)$
Solution
Note that tan and sec and their derivative are defined for $cos(x) \neq 0$, thus $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$. Thus the domain of definition of f and f' is the set of all real number x with $x \neq \pi/2 + k\pi$, for any $k \in \mathbb{Z}$.
$$f'(x) = 3sec^2(x) - sec(x)tan(x)$$

10. $f(x) = \frac{-1}{sin(x)} + 5cot(x)$
Solution
Note that $cot(x) = \frac{1}{tan(x)}$ thus cot is defined when tan is defined that is for $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$ as seen in the previous question and when $tan(x) \neq 0$ that is $sin(x) \neq 0$ thus $x \neq \pi l$, $l \in \mathbb{Z}$. In conclusion cot is defined when $x \neq \pi/2 + k\pi/2$, $k \in \mathbb{Z}$ and since $x \mapsto 1/sin(x)$ is also define when $x \neq \pi/2 + k\pi/2$, $k \in \mathbb{Z}$ the domain of definition of f and f' is the set of all real values with $x \neq \pi/2 + k\pi/2$ for any $k \in \mathbb{Z}$
Note that
$$f(x) = -csc(x) + 5cot(x)$$
Thus
$$f'(x) = csc(x)cot(x) - 5csc(x)^2 = csc(x)(cot(x) - 5csc(x))$$

11. $f(x) = (1 + 2x + 3x^2)(5x^5 - 4x^4)$
Solution f is a product of two polynomials thus the domain of definition of f and f' is $\mathbb{R}$.

We write $f(x) = u(x)v(x)$, with $u(x) = 1+2x+3x^2$ thus $u'(x) = 6x+2$ and $v(x) = 5x^5 - 4x^4 = x^4(5x-4)$ thus $v'(x) = 25x^4 - 16x^3 = x^3(25x - 16)$.
Using the product rule we get:
$$\begin{aligned} f'(x) &= u'(x)v(x) + v'(x)u(x) = x^4(5x-4)(6x+2) + x^3(25x-16)(1+2x+3x^2) \\ &= x^3(x(5x-4)(6x+2) + (25x-16)(1+2x+3x^2)) \\ &= x^3(105x^3 - 12x^2 - 15x - 16) \end{aligned}$$

12. $f(t) = (\frac{1}{t^3} - \frac{1}{t})cos(t)$
f is product of a rational function with a cos function thus defined and differentiable for $t \neq 0$. Using the product rule and power rule we get:
$$f'(t) = -(\frac{1}{t^3} - \frac{1}{t})sin(t) + (\frac{-3}{t^4} + \frac{1}{t^2})cos(t)$$

13. $f(\theta) = \theta cot(\theta) - \theta^2 cos(\theta)$
   f is a product sum of trigonometric function with polynomial. All of them are defined and differnetiable over $\mathbb{R}$ except for *cot* that is defined and differentiable for $x \neq \pi/2 + \pi/2k$, $k \in \mathbb{Z}$. Thus the domain of definition of f and f' is $x \neq \pi/2 + \pi/2k$, for any $k \in \mathbb{Z}$.
   Using the product rule we get:
   $$f'(\theta) = -\theta csc(\theta)^2 + cot(\theta) - 2\theta cos(\theta) + \theta^2 sin(\theta)$$

14. $f(x) = \frac{x^3+1}{3-x}$
   f is a rational function thus define and differentiable when its denominator is non zero that is when $3 - x \neq 0$ that is $x \neq 3$.
   Note that
   $$f(x) = \frac{u(x)}{v(x)}$$
   where $u(x) = x^3 + 1$ thus $u'(x) = 3x^2$ and $v(x) = 3 - x$ thus $v'(x) = -1$.
   Then, using the quotient rule we get
   $$f'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)} = \frac{3x^2(3-x) + (x^3+1)}{(3-x)^2} = \frac{-2x^3 + 9x^2 + 1}{(3-x)^2}$$

15. $f(t) = \frac{1}{1-4t^{-2}}$
   Solution: f is a rational function and is defined and differentiable when the denominator is non zero that is $t \neq 0$ and
   $$1 - 4t^{-2} \neq 0 \Leftrightarrow t^2 - 4 \neq 0 \Leftrightarrow t \neq \pm 2$$
   Then, using the quotient and power rule we get
   $$f'(t) = \frac{8t^{-3}}{1 - 4t^{-2}}$$

16. $f(x) = \frac{x}{x^2-4} - \frac{x-1}{x^2+4}$
   Solution: f is a sum of two rational functions thus defined and differentiable when their denominator is nonzero that is $x^2 - 4 \neq 0$ and $x^2 + 4$ this is equivalent to $x \neq \pm 2$.
   Then, using the quotient and power rules we get
   $$f'(x) = \frac{x^2 - 4 - 2x^2}{(x^2-4)^2} = \frac{-x^2 - 4}{(x^2-4)^2}$$

17. $f(y) = \frac{\sqrt{y} - sin(y)}{tan(y) + y^5}$
   Solution: f is the quotient of two function with a numerator involving a square root function. Thus f is defined when $y \geq 0$ and $tan(y) + y^5 \neq 0$.
   $$f'(x) = \frac{(\frac{1}{2\sqrt{y}} + cos(y))(tan(y) + y^5) - (\sqrt{y} - sin(y))(sec^2(y) + 5y^4)}{(tan(y) + y^5)^2}$$
   As a consequence, f is differentiable when $y > 0$ and $tan(y) + y^5 \neq 0$.

18. $f(x) = \frac{x}{x+3/x}$
   Solution:
   f is a rational function thus defined and differentiable when $x \neq 0$ for $3/x$ to be defined and $x + 3/x \neq 0$ that is $x^2 \neq -3$ but x^2 can never be equal to -3. Thus the domain of definition of f and f' is $(-\infty, 0) \cup (0, \infty)$.
   $$f'(x) = \frac{x + 3/x - x(1 - 3/x^2)}{(x+3/x)^2} = \frac{3/x(1 + 3/x)}{(x+3/x)^2}$$

19. $f(x) = \frac{x^2 cot(x)}{x+1}$
Solution: f is a quotient of functions thus is defined and differentiable when cot is defined that is $x \neq \pi/2 + \pi/2k$, for any $k \in \mathbb{Z}$ and $x + 1 \neq 1 \Leftrightarrow x \neq -1$.
Using the quotient rule we get:
$$f'(x) = \frac{(-x^2 csc^2(x) + 2xcot(x))(x+1) - x^2 cot(x)}{(x+1)^2}$$

20. $f(x) = xsin(x) + \frac{x}{cos(x)}$
Solution : f is sum of product and quotient of polynomials and trigonometric functions. Thus f is defined and differentiable when $cos(x) \neq 0$ that is $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$.
Using the product and quotient rules, we get:
$$f'(x) = sin(x) + xcos(x) + \frac{cos(x) + xsin(x)}{cos(x)^2}$$

21. $f(x) = \frac{\sqrt{x^3}}{sec(x)(2x+3)}$
f is a quotient of a square root function by the product of a trigonometric function and a polynomial function. f is thus defined when $\sqrt{x^3}$ is defined that is $x \geq 0$, sec is defined that is $x \neq \pi/2 + \pi k$, where $k \in \mathbb{Z}$ and $2x + 3 \neq 0$ thus $x \neq -2/3$.
Then, using the quotient and product rules we get:
$$f'(x) = \frac{3/2\sqrt{x}sec(x)(2x+3) - \sqrt{x^3}(sec(x)tan(x)(2x+3) + 2sec(x))}{(sec(x)(2x+3))^2}$$
f' is thus defined for any real values x such that $x \geq 0$, $x \neq \pi/2 + \pi k$, where $k \in \mathbb{Z}$ and $x \neq -2/3$.

22. $f(t) = \frac{t^2+1}{(2t-3)^2}$
Solution: f is a rational function thus defined and differentiable for any real values such that it denominator is nonzero that is $2t - 3 \neq 0 \Leftrightarrow t \neq 3/2$.
Thus using the quotient and chain rules, we get:
$$\begin{aligned} f'(t) &= \tfrac{2t(2t-3)^2 - 4(2t-3)(t^2+1)}{(2t-3)^4} \\ &= \tfrac{(2t-3)(2t(2t-3) - 4(t^2+1))}{(2t-3)^4} \\ &= \tfrac{-6t-4}{(2t-3)^3} \end{aligned}$$

23. $f(x) = \sqrt{x^2 - 3x^{-1} + 5}$
Solution: f is a square root function of a rational function thus defined for any real values x such that $x^2 - 3x^{-1} + 5 \geq 0 \Leftrightarrow x^3 + 5x - 3 \geq 0$ (here we would need at least to know one root to go further) and $x \neq 0$.
Thus, using the chain rule we get:
$$f'(x) = (2x + 3x^{-2})\frac{1}{2\sqrt{x^2 - 3x^{-1} + 5}}$$
We observe that f is differentiable when f was defined.

24. $f(x) = (2sin(x) - 3x^9)^{25}$
Solution: f is a power of a sum of a trigonometric with a power function. Thus f is defined and differentiable for any real values.
Thus using the power and chain rules, we get
$$f'(x) = 25(2cos(x) - 27x^8)(2sin(x) - 3x^9)^{24}$$

25. $f(x) = csc(3x - 1)$
Solution: f is a composite of a trigonometric function with a polynomial function thus define and differentiable for any real values such that when $sin(3x - 1) \neq 0$ that is $3x - 1 \neq \pi k$, $k \in \mathbb{Z}$ that is also equivalent to $x \neq \frac{\pi k + 1}{3}$, $k \in \mathbb{Z}$.
Thus using the chain rule we get
$$f'(x) = -3csc(3x - 1)cot(3x - 1)$$

26. $f(r) = \pi tan(\pi r^2 - 5r)$

Solution f is a trigonometric function thus defined and differentiable for any real values such that $cos(\pi r^2 - 5r) \neq 0$ that is for any $k \in \mathbb{Z}$, $\pi r^2 - 5r \neq \pi/2 + \pi k \Leftrightarrow \pi r^2 - 5r - \pi/2 - \pi k \neq 0$, the discriminant of those polynomial is $\Delta_k = 25 + 4(\pi/2 + \pi k)$, when $\Delta_k < 0$ there is no such r, when $\Delta_k \geq 0$ that is $k \geq \frac{-25-2\pi}{4\pi} \sim -2.49$ we have two such r for each values of k that is $\frac{5 \pm \sqrt{\Delta_k}}{2}$. Thus f is defined and differentiable for any real value x such that $x \neq \frac{5\pm\sqrt{\Delta_k}}{2}$, where $25 + 4(\pi/2 + \pi k)$ and $k \geq -2$.

Thus using the chain rule we get:

$$f'(r) = \pi(2\pi r - 5)sec^2(\pi r^2 - 5r)$$

27. $f(x) = (1 + (2x + 1)^6)^7$

Solution: f is a polynomial function thus defined and differentiable for any real values.

Thus using the chain rule twice we get:

$$f'(x) = 84(2x + 1)^5(1 + (2x + 1)^6)^6$$

28. $f(x) = sin^3(x^2)$

Solution: f is the composite of polynomials and trigonometric function defined and differentiable for any real values x. Thus f is also defined and differentiable for any real values.

Here I chose to use the careful method just because I need to think about two much things (use it as soon as it becomes for you too tricky to do everything at once) but also use it if you have a tendency to do a lot of computation mistake you can use this method in any of the example above.

We observe that $f(x) = u(v(x))$ where $u(x) = x^3$ thus $u'(x) = 3x^2$ and $v(x) = sin(x^2)$ **(Be careful: this is a composite of two functions too...)** thus using the chain rule, $v'(x) = 2xcos(x^2)$, then

Thus using the chain rule twice we get

$$f'(x) = v'(x)u'(v(x)) = 6xcos(x^2)sin(x^2)^2$$

29. $f(x) = \sqrt{cos(x^2 - 3x^{-8} + 1)}$

Solution: f is the square root of a composite of a trigonometric with a polynomial function. Thus define when $cos(x^2 - 3x^{-8} + 1) \geq 0$ that is $-\pi/2 + 2\pi k \leq x^2 - 3x^{-8} + 1 \leq \pi/2 + 2\pi k$ for $k \in \mathbb{Z}$ and differentiable for any real values such that $-\pi/2 + \pi k < x^2 - 3x^{-8} + 1 < \pi/2 + \pi k$. (we cannot really say much more here.)

Here I have to deal with a composite of a composite too much to think about in my head at once thus I prefer to be careful again.

Note that $f(x) = u(v(x))$ where $u(x) = \sqrt{x}$ thus $u'(x) = \frac{1}{2\sqrt{x}}$ and $v(x) = cos(x^2 - 3x^{-8} + 1)$ thus using the chain rule, $v'(x) = -(2x - 24x^{-9})sin(x^2 - 3x^{-8} + 1)$.

Thus using the chain rule twice, we get

$$f'(x) = v'(x)u'(v(x)) = -(2x - 24x^{-9})sin(x^2 - 3x^{-8} + 1)\frac{1}{2\sqrt{cos(x^2 - 3x^{-8} + 1)}}$$

30. $f(x) = tan(cot(x^2))$

Solutions: f is the composite of trigonometric functions and polynomial function and is defined and differentiable for any real value x such that $cos(cot(x^2)) \neq 0$ that is $cot(x^2) \neq \pi/2 + \pi k$ with $k \in \mathbb{Z}$ **(that is complicated to describe thus we will not go further)** and $x^2 \neq \pi/2 + \pi/2k$ for $k \in \mathbb{Z}$ that is $x \neq \pm\sqrt{\pi/2 + \pi/2k}$ for k integer $k \geq -1$.

Here again composite of composite I am again careful

We observe that $f(x) = u(v(x))$ where $u(x) = tan(x)$ thus $u'(x) = sec^2(x)$ and $v(x) = cot(x^2)$ thus using the chain rule, $v'(x) = -2xcsc^2(x^2)$.

Then, using the chain rule again we get,

$$f'(x) = v'(x)u'(v(x)) = -2xcsc^2(x^2)sec^2(cot(x^2))$$

31. $f(x) = sin^2(cos^2(x))$

Solution: f is the composite of trigonometric function and power function. Thus defined and differentiable for any real values.

Here I have the composite of a composite of a composite of a composite ai ai ai, ok lets try to be careful.

Observe that $f(x) = u(v(x))$ where $u(x) = x^2$ thus $u'(x) = 2x$ and $v(x) = sin(cos^2(x)) = w(z(x))$ where $w(x) = sin(x)$ thus $w'(x) = cos(x)$ and $z(x) = cos^2(x)$ and thus applying the chain rule $z'(x) = -2sin(x)cos(x) = -sin(2x)$, therefore, applying also the chain rule we get $v'(x) = z'(x)w'(z(x)) = -sin(2x)cos(cos^2(x))$.

We finally apply the chain rule one last time and get:

$$f'(x) = v'(x)u'(v(x)) = -2sin(2x)cos(cos^2(x))sin(cos^2(x)) = -sin(2x)sin(2cos^2(x))$$

32. $f(x) = sec(3x)csc(5x)$

Solution: f is a product of a composite of trigonometric function with polynomial function. Thus f is defined and differentiable for any value of x such that $cos(x) \neq 0$ and $sin(x) \neq 0$ that is $x \neq \pi/2 + \pi/2k$ where $k \in \mathbb{Z}$.

Thus, using the product rule and twice the chain rule, we get

$$f'(x) = 3sec(3x)tan(3x)csc(5x) - 5sec(3x)csc(5x)cot(5x)$$

33. $f(x) = (3x^2 - 9)^{17}(x^4 - x^2 + 1)^{31}$

Solution: f is a polynomial function thus defined and differentiable for any real values.

Using the product rule and twice the chain rule we get:

$$\begin{aligned} f'(x) &= 31(3x^2-9)^{17}(4x^3-2x)(x^4-x^2+1)^{30} + 102x(3x^2-9)^{16}(x^4-x^2+1)^{31} \\ &= 6x(3x^2-9)^{16}(x^4-x^2+1)^{30}(31(x^2-3)(2x^2-1) + 17(x^4-x^2+1)) \\ &= 6x(3x^2-9)^{16}(x^4-x^2+1)^{30}(79x^4-234x^2+110) \end{aligned}$$

34. $f(x) = \frac{sin(x^2)}{cos^2(x)}$

Solution: f is a quotient of trigonometric function, thus f is defined and differentiable for any real values x such that $cos(x) \neq 0$ that is $x \neq \pi/2 + \pi k$, $k \in \mathbb{Z}$.

Here I prefer to be careful too much risk to make mistakes

Observe that $f(x) - \frac{u(x)}{v(x)}$ where $u(x) = sin(x^2)$ thus applying the chain rule we get $u'(x) = 2xcos(x^2)$ and $v(x) = cos^2(x)$ thus applying the chain rule $v'(x) = -2sin(x)cos(x) = -sin(2x)$.

Using the quotient rule we get

$$f'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)} = \frac{2xcos(x^2)cos^2(x) + sin(2x)sin(x^2)}{cos^4(x)}$$

35. $f(x) = (x^3 - 8)tan^2(5x - 3)$

Solution: f is the product of a polynomial with a composite of polynomials with trigonometric function.

Here I decide to be careful again, why take a risk. Remember I always give you some credit if you write the correct formula even if sometimes you make a computational mistake.

We observe that $f(x) = u(x)v(x)$ where

$u(x) = x^3 - 8$ thus $u'(x) = 3x^2$ and

$v(x) = tan^2(5x - 3) = w(z(x))$ whit $w(x) = x^2$ thus $w'(x) = 2x$ and $z(x) = tan(5x - 3)$ thus using the chain rule, we get $z'(x) = 5sec^2(5x - 3)$. Therefore, using the chain rule again we get $v'(x) = z'(x)w'(z(x)) = 10sec^2(5x - 3)tan(5x - 3)$.

Finally using the product rule we get

$$f'(x) = u'(x)v(x) + v'(x)u(x) = 3x^2tan^2(5x - 3) + 10sec^2(5x - 3)tan(5x - 3)(x^3 - 8)$$

36. $f(x) = \frac{(3x^3+x)^4}{(\sqrt{x}-x^{-2})^5}$
Solution: f is a quotient of a polynomial by a power of a sum of power function wit a square root function. f is then defined for any real value such that $x \geq 0$ and $\sqrt{x} - x^{-2} \neq 0$ that is $\sqrt{x}(1 - x^{-5/2}) \neq 0 \leftrightarrow x \neq 0$ and $x^5 \neq 1$ which turn out to be equivalent to $x \neq 0$ and $x \neq \pm 1$.
Using the quotient, twice the chain rule and power rules we have:

$$\begin{aligned} f'(x) &= \frac{4(9x^2+1)(3x^3+x)^3(\sqrt{x}-x^{-2})^5 - 5(\frac{1}{2\sqrt{x}}+2x^{-3})(\sqrt{x}-x^{-2})^4(3x^3+x)^4}{(\sqrt{x}-x^{-2})^{10}} \\ &= \frac{(3x^3+x)^3(\sqrt{x}-x^{-2})^4(4(9x^2+1)(\sqrt{x}-x^{-2}) - 5(\frac{1}{2\sqrt{x}}+2x^{-3})(3x^3+x))}{(\sqrt{x}-x^{-2})^{10}} \\ &= \frac{-1/2(3x^3+x)^3(\sqrt{x}-x^{-2})^4(-57x^5+132x^{5/2}-3x^3+28\sqrt{x})}{x^{5/2}(\sqrt{x}-x^{-2})^{10}} \end{aligned}$$

Observe that f' as same domain of definition as f, for the same reason as above.

37. $f(x) = \left(\frac{6x-x^4}{sin(x)}\right)^6$
Solution: f is a composite of a power with a quotient of a polynomial by a trigonometric function, it is thus defined and differentiable for any real values such that $sin(x) \neq 0$ that is $x \neq \pi k$ where $k \in \mathbb{Z}$.
writing the above sentence will help you to compute the derivative correctly, by already knowing which rule can apply
Using the chain rule and the quotient rule we get

$$\begin{aligned} f'(x) &= 6\left(\frac{(6-4x^3)sin(x)-(6x-x^4)cos(x)}{sin^2(x)}\right)\left(\frac{6x-x^4}{sin(x)}\right)^5 \\ &= \frac{((6-4x^3)sin(x)-(6x-x^4)cos(x))(6x-x^4)^5}{sin^7(x)} \end{aligned}$$

38. $f(x) = 3e^x - 7log_{10}(x)$
Solution: f is a sum of an exponential function with a log function thus defined and differentiable for any real values x such that $x > 0$.
Using exponential and logarithm differentiation rule we get:

$$f'(x) = e^x - 7\frac{1}{ln(10)x}$$

39. $f(x) = ln(x^2+1) - log_2(5x)$
Solution: f is a difference of composites of a logarithm function with a polynomial function thus is defined and differentiable when $5x > 0$ that is $x > 0$ and $x^2 + 1 > 0$ that is always true. Thus the domain of definition of f and f' is $(0, \infty)$.
Using the log derivative rules and chain rules we get;

$$f'(x) = \frac{2x}{x^2+1} - \frac{5}{5ln(2)x} = \frac{2x}{x^2+1} - \frac{1}{ln(2)x}$$

40. $f(x) = e^{x^2} \cdot ln(tan(x))$
Solution: f is the product of a composite of an exponential function with a polynomial function and a composite of ln function with tan. Thus f is defined and differentiable for each real values x such that $tan(x) > 0$ that is $k\pi < x < \pi/2 + \pi k$ where $k \in \mathbb{Z}$.
Using the product rule and the chain rule twice we get :

$$f'(x) = e^{x^2}\frac{sec^2(x)}{tan(x)} + 2xe^{x^2}ln(tan(x))$$

41. $f(z) = \frac{2^z - z^2}{1-log_3(z)}$
Solution: f is the quotient of a polynomial with a log function. Thus f is defined and differentiable for each values z such that $z > 0$ and $1 - log_3(z) \neq 0$ that is $log_3(z) \neq 1$ which turn out to be equivalent to $z \neq 3$. Thus the domain of definition of f and f' is $(0, 3) \cup (3, \infty)$.
Remember $2^z = e^{ln(2)z}$!!
Using the quotient, exponential and logarithm rules we get:

$$f'(x) = \frac{(ln(2)2^z - 2z)(1 - log_3(z)) - \frac{2^z-z^2}{ln(3)z}}{(1 - log_3(z))^2}$$

42. $f(x) = e^{xsec(x)} + e^5$
Solution: f is the sum of a composite of a exponential function with a product of a polynomial with a trigonometric function and a constant. e^5 **is a constant !!!!** Thus f is defined and differentiable for any x values such that sec is defined that is $x \neq \pi/2 + \pi k$ where $k \in \mathbb{Z}$.
Using the chain rule and product rule we get:

$$f'(x) = (xsec(x)tan(x) + sec(x))e^{xsec(x)} = sec(x)(xtan(x) + 1)e^{xsec(x)}$$

43. $f(x) = \frac{3^{-sin(x)}}{1+ln(x^3-x)}$
Solution: f is a quotient of a composite of an exponential with a trigonometric function and sum of a constant with a composite of a logarithm with a polynomial function. Thus f is define and differentiable for any real value x such that $1 + ln(x^3 - x) \neq 0$ that is $ln(x^3 - x) \neq -1$ that is $x^3 - x \neq e^{-1}$ **here it is hard to find x satisfying this cubic equation thus we stop here** and $x^3 - x > 0$ that is $x(x^2 - 1) > 0$ using a sign table for instance we find out that this is satisfied over $(-1, 0) \cup (1, \infty)$.
Using the chain rule twice and the quotient rule we get

$$\begin{aligned} f'(x) &= \frac{-ln(3)cos(x)3^{-sin(x)}(1+ln(x^3-x))+3^{-sin(x)}\frac{3x^2-1}{x^3-x}}{(1+ln(x^3-x))^2} \\ &= \frac{3^{-sin(x)}(-ln(3)cos(x)(1+ln(x^3-x))+\frac{3x^2-1}{x^3-x})}{(1+ln(x^3-x))^2} \end{aligned}$$

I need to see your work mean the details on when using chain rules, product and quotient, if you do not show no credit for the question... Avoid to develop rather factorize as much as you can if possible into a product of easy sign determination function.

44. $f(x) = ln(\frac{x^5}{(2x-1)^3(x^2+1)})$
Solution: f is a composite of ln with a rational function. Thus f is defined and differentiable for any real value x such that $(2x - 1)^3(x^2 + 1) \neq 0$ that is $x \neq 1/2$ and $\frac{x^5}{(2x-1)^3(x^2+1)} > 0$ using a sign table for instance we get that this is satisfied over $(-\infty, 0) \cup (1/2, \infty)$. Thus the domain of definition of f and f' is $(-\infty, 0) \cup (1/2, \infty)$.
Note that using the property of the logarithm we have that

$$f(x) = ln(\frac{x^5}{(2x-1)^3(x^2+1)}) = 5ln(x) - 3ln(2x - 1) - ln(x^2 + 1)$$

Thus, applying two chain rule we get

$$f'(x) = \frac{5}{x} - 3\frac{2x}{2x-1} - \frac{2x}{x^2+1} = \frac{5}{x} - \frac{6x}{2x-1} - \frac{2x}{x^2+1}$$

45. $f(x) = \frac{(3x^3-1)^4}{(x^4+x)^{(2x-1)}}$
Solution: $f$is a quotient of a polynomial by a composite of a power function and polynomial function. Thus f is define when $x^4 + x = x(x^3 + 1) > 0$ that is doing a sign table over $(-\infty, -1) \cup (0, \infty)$.
Here I chose the careful way and will first rewrite the function a bit

$$f(x) = \frac{(3x^3-1)^4}{e^{ln(x^4+x)(2x-1)}} = \frac{u(x)}{v(x)}$$

where $u(x) = (3x^3 - 1)^4$ thus using the chain rule we get $u'(x) = 36x^2(3x^3 - 1)^3$ and $v(x) = e^{ln(x^4+x)(2x-1)}$ thus using the chain and product rule we get $v'(x) = (2ln(x^4 + x) + \frac{(2x-1)(3x^3+1)}{x^4+x})e^{ln(x^4+x)(2x-1)}$
Thus using the quotient rule we get

$$\begin{aligned} f'(x) &= \frac{u'(x)v(x)-v'(x)u(x)}{v(x)^2} \\ &= \frac{36x^2(3x^3-1)^3e^{ln(x^4+x)(2x-1)}-(2ln(x^4+x)+\frac{(2x-1)(3x^3+1)}{x^4+x})e^{ln(x^4+x)(2x-1)}(3x^3-1)^4}{(x^4+x)^{2(2x-1)}} \\ &= \frac{(3x^3-1)^3e^{ln(x^4+x)(2x-1)}(36x^2-(2ln(x^4+x)+\frac{(2x-1)(3x^3+1)}{x^4+x})(3x^3-1))}{(x^4+x)^{2(2x-1)}} \end{aligned}$$

46. $f(x) = (sin(x))^{cot(x)}$
Solution: f is a composite of trigonometric function and exponentiable function thus defined and differentiable for any real values x such that $x \neq \pi/2 + \pi/2k$, $k \in \mathbb{Z}$ and $sin(x) > 0$ that is $2\pi k < x < \pi + 2\pi k$, $k \in \mathbb{Z}$.
Noting that
$$f(x) = (sin(x))^{cot(x)} = e^{ln(sin(x))cot(x)}$$
Using the chain rule and the product rule we get:
$$f'(x) = (\frac{cos(x)}{sin(x)}cot(x) - ln(sin(x))csc^2(x))e^{ln(sin(x))cot(x)}$$

47. $f(x) = \sqrt[3]{2x-5}$
Solution: f is the composite of a cube root with a polynomial function thus f is defined and differentiable for any real value.
Using the chain rule and the power rule we get:
$$f'(x) = 2/3(2x-5)^{-2/3}$$

48. $f(x) = \sqrt[3]{2 + tan(x^2)}$
Solution: f is the composite of cube root and a trigonometric function thus it is defined when $cos(x^2) \neq 0$ that is $x^2 \neq -\pi/2 + \pi k$ where $k \in \mathbb{Z}$ that is $x \neq \pm\sqrt{-\pi/2 + \pi k}$ when $k \geq 0$.
Using the chain rule twice we obtain:
$$f'(x) = 2/3xsec^2(x^2)(2 + tan(x^2))^{-2/3}$$

49. $f(x) = x^3(5x^2+1)^{-2/3}$
Solution: f is the quotient of a polynomial by a cube root of a polynomial thus f is defined and differentiable for any real value x such that $5x^2 + 1 \neq 0$ but $5x^2 + 1$ is never x thus the domain of definition of f and f' is $\mathbb{R}$.

2. Find the derivative $y' = dy/dx$ in each case by using implicit differentiation.

1. $y^3 - y + cos(x) + sin(y) = x^{10}$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$3y'y^2 - sin(x) + 2y'cos(y) = 10x^9$$
That is
$$y'(3y^2 + 2cos(y)) = 10x^9 + sin(x)$$
We then obtain
$$y' = \frac{10x^9 + sin(x)}{3y^2 + 2cos(y)}$$

2. $x^2y + xy^2 = x^2 + 1$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$2xy + x^2y' + y^2 + 2y'y = 2x$$
That is
$$y'(x^2 + 2y) = 2x - 2xy - y^2$$
We then obtain that
$$y' = \frac{2x - 2xy - y^2}{x^2 + 2y}$$

3. $\frac{x^2+ycos(x)+3}{y^2-1} = 5$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
Note that
$$x^2 + ycos(x) + 3 = 5(y^2 - 1)$$
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$2x + y'cos(x) - ysin(x) = 10y'y$$
That is
$$y'(10y - cos(x)) = 2x - ysin(x)$$
We then obtain that
$$y' = \frac{2x - ysin(x)}{10y - cos(x)}$$

4. $ye^x + 2log_2(x) - y^2 = ln(x)$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$y'e^x + ye^x + \frac{2}{ln(2)x} - 2y'y = 1/x$$
That is,
$$y'(e^x - 2y) = 1/x - ye^x - \frac{2}{ln(2)x}$$
Thus
$$y' = \frac{1/x - ye^x - \frac{2}{ln(2)x}}{(e^x - 2y)}$$

5. $sin(y^2) - cos(xy) = 1$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$2y'cos(y^2) + (xy' + y)sin(xy) = 0$$
That is,
$$y'(2cos(y^2) + xsin(xy)) = -ysin(xy)$$
Thus we obtain
$$y' = \frac{-ysin(xy)}{2cos(y^2) + xsin(xy)}$$

6. $e^{x^2+y^2} = x^2 + y^2$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:
$$(2x + 2y'y)e^{x^2+y^2} = 2x + 2yy'$$
That is
$$y'(2ye^{x^2+y^2} - 2y) = 2x - 2xe^{x^2+y^2}$$
Thus, we obtain
$$y' = \frac{2x(1 - e^{x^2+y^2})}{2y(e^{x^2+y^2} - 1)} = -\frac{x}{y}$$

7. $x + xy - 2x^3 = 2$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$1 + xy' + y - 6x^2 = 0$$

That is ,

$$xy' = -1 - y + 6x^2$$

Thus we obtain

$$y' = \frac{-1 - y + 6x^2}{x}$$

8. $x^3 + y^3 = 3xy^2$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$3x^2 + 3y'y^2 = 3(2xy'y + y^2)$$

That is

$$y'(3y^2 - 6xy) = 3(y^2 - x^2)$$

Thus we obtain that

$$y' = \frac{y^2 - x^2}{y^2 - 2xy}$$

9. $x^2y + 3xy^3 - x = 3$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$2xy + x^2y' + 3y^3 + 9xy'y^2 - 1 = 0$$

That is,

$$y'(x^2 - 9xy^2) = 1 - 2xy - 3y^3$$

Thus we obtain:

$$y' = \frac{1 - 2xy - 3y^3}{x(x - 9y^2)}$$

10. $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$-1/2x^{-3/2} - 1/2y'y^{-3/2} = 0$$

That is

$$1/2y'y^{-3/2} = -1/2x^{-3/2}$$

Thus we obtain

$$y' = \frac{-1/2x^{-3/2}}{1/2y^{-3/2}} = \frac{-x^{-3/2}}{y^{-3/2}}$$

11. $sin(x^2y^2) = x$
Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!
We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$(2xy^2 + 2x^2y'y)cos(x^2y^2) = 1$$

That is,

$$2x^2y'ycos(x^2y^2) = 1 - 2xy^2cos(x^2y^2)$$

Thus we obtain

$$y' = \frac{1 - 2xy^2cos(x^2y^2)}{2x^2y'ycos(x^2y^2)}$$

3. Find where the tangent line is horizontal for each function.

   Solution: A line is horizontal if and onl it slope is 0. Also we know that the slope of a tangent at x is given by $f'(x)$. Thus the question is asking us in the 3 cases to find all the real values x such that $f'(x) = 0$.

   1. $f(x) = 9x^4 - 4x^3 - 48x^2$
      Using the power rules, we get $f'(x) = 36x^3 - 12x^2 - 96x$. Thus we are trying to find x such that

      $$f'(x) = 0$$

      That is

      $$x(36x^2 - 12x - 96) = 0$$

      Thus we obtain either $x = 0$ or $36x^2 - 12x - 96 = 0$ the discriminant of this polynomial is $(-12)^2 - 4(-96)36 = 13968$ and $x = \frac{12 \pm \sqrt{13968}}{72}$.

      Thus the tangent line to the graph of f at x is horizontal is and only if $x = 0$ and $x = \frac{12 \pm \sqrt{13968}}{72}$.

   2. $f(x) = (2x + 3)e^{x^2}$
      Using the product and chain rule we get:

      $$f'(x) = 2e^{x^2} + 2x(2x + 3)e^{x^2} = (4x^2 + 6x + 2)e^{x^2} = 2(2x^2 + 3x + 1)e^{x^2}$$

      Thus $f'(x) = 0$ if and only if $(4x^2 + 6x + 2)e^{x^2} = 0$ but since $2e^{x^2} > 0$ that is equivalent to $2x^2 + 3x + 1 = 2(x + 1)(x + 1/2) = 0$ Thus, is equivalent to $x = -1$ or $-1/2$.

      Thus the tangent line to the graph of f at x is horizontal is and only if $x = -1$ and $x = -1/2$.

   3. $\sqrt{y} + xlog_2(x) - 4x = cot(y) + \frac{x}{ln(2)}$
      Here we first use implicit differentiation to find y'. We differentiate with respect to x thinking of y as a function of x, we get:

      $$1/2y'y^{-1/2} + log_2(x) + \frac{x}{ln(2)x} - 4 = -y'csc^2(y) + \frac{1}{ln(2)}$$

      That is,

      $$y'(1/2y^{-1/2} + csc^2(y)) = \frac{1}{ln(2)} - log_2(x) - \frac{1}{ln(2)} + 4$$

      Thus we obtain

      $$y' = \frac{-log_2(x) + 4}{1/2y^{-1/2} + csc^2(y)}$$

      We thus get that $y' = 0$ if and only $log_2(x) = 4$ (a quotient is 0 if and only if its numerator is 0) that is equivalent to $x = 2^4 = 16$.

      Thus the tangent line to the graph of f at x is horizontal is and only if $x = 16$.

4. Find the required higher derivative in each case.

   1. Find $f''(x)$ for $f(x) = tan(x^3)$
      Solution: Using the chain rule we obtain that

      $$f'(x) = 3x^2sec^2(x^3)$$

      Using the chain rule and the product rule we get:

      $$\begin{aligned} f''(x) &= 6xsec^2(x^3) + 3x^2 \times (2 \times 3x^2sec(x^3)tan(x^3)sec(x^3)) \\ &= 6xsec^2(x^3)(1 + 3x^3tan(x^3)) \end{aligned}$$

2. Find $f'''(x)$ for $f(x) = ln(x^2+1)$
Solution: Using the chain rule we obtain.

$$f'(x) = \frac{2x}{x^2+1}$$

Using the quotient rule we get

$$f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

Using again the quotient and the chain rule we obtain:

$$f'''(x) = \frac{-4x(x^2+1)^2 - 4x(x^2+1)(-x^2+1)}{(x^2+1)^4} = \frac{-8x}{(x^2+1)^3}$$

3. Find $f^{(365)}(x)$, for $f(x) = log_5(3x)$.
Here we will compute a few and then try to make a conjecture and then prove it.
Using the logarithm rule we get:

$$f'(x) = \frac{1}{3ln(5)x} = \frac{1}{3ln(5)}x^{-1}$$

Using the power rule we obtain:

$$f''(x) = -\frac{1}{3ln(5)}x^{-2}$$

$$f'''(x) = 2\frac{1}{3ln(5)}x^{-3}$$

$$f^{(4)}(x) = 2 \times (-3)\frac{1}{3ln(5)}x^{-4}$$

$$f^{(5)}(x) = 2 \times (-3) \times (-4)\frac{1}{3ln(5)}x^{-5}$$

It seems that

$$f^n(x) = (-1)^{n+1}(n-1)!\frac{1}{3ln(5)}x^{-n}$$

We will prove this by induction on n.
We initialize at $n = 1$, and see that indeed the formula is correct since

$$f'(x) = \frac{1}{3ln(5)}x^{-1}$$

We now suppose that the formula is true for an arbitrary natural integer n that is

$$f^n(x) = (-1)^{n+1}(n-1)!\frac{1}{3ln(5)}x^{-n}$$

and prove that the formula remains true for $n+1$ that is

$$f^{n+1}(x) = (-1)^{n+2}n!\frac{1}{3ln(5)}x^{-n-1}$$

But

$$f^{n+1}(x) = f^{n\prime}(x) = -(-1)^{n+1}(n-1)!n\frac{1}{3ln(5)}x^{-n-1} = (-1)^{n+2}n!\frac{1}{3ln(5)}x^{-n-1}$$

We have thus prove by induction that this formula is true for any natural number n and thus

$$f^{(365)}(x) = 364!\frac{1}{3ln(5)}x^{-365}$$

5. Find $y'' = d^2y/d^2x$ using implicit differentiation.

   (a) $2x^2 - 3y^2 = 4$

   Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!

   We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

   $$4x - 3y'y = 0$$

   That is

   $$y' = \frac{4x}{3y}$$

   Then, applying implicit differentiation again as before to the last equality we obtain:

   $$y'' = \frac{12y - 12y'}{9y^2} = \frac{12y - 12\frac{4x}{3y}}{9y^2} = \frac{4y - 4\frac{4x}{3y}}{3y^2} = \frac{12y^2 - 16x}{3y^3}$$

   (b) $xy + y^2 = 2$

   Solution: BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!

   We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

   $$y + xy' + 2yy' = 0$$

   That is

   $$y'(x + 2y) = -y$$

   Thus we obtain

   $$y' = \frac{y}{x + 2y}$$

   Then, applying implicit differentiation again as before to the last equality we obtain:

   $$y'' = \frac{y'(x + 2y) - (1 + 2y')y}{(x + 2y)^2} = \frac{y'x - y}{(x + 2y)^2} = \frac{\frac{y}{x+2y}x - y}{(x + 2y)^2} = \frac{yx - y(x + 2y)}{(x + 2y)^3} = \frac{-2y^2}{(x + 2y)^3}$$

6. Use implicit differentiation to find the slope of the tangent line to the curve at the specified point

   Solution: The slope of the tangent at a point with abscissa x is equal to the derivative of the function at this point. So in order to do this exercise we will consider y to be a function of x and do an implicit differentiation, so that we can find the differiantiation of y at the point x and then compute y' for x at the point specified.

   (a) $x^4 + y^4 = 16$ at the point $(1, \sqrt[4]{15})$;

   BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!

   We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

   $$4x^3 + 4y'y^3 = 0$$

   Thus we obtain

   $$y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3}$$

   Then at the point $(1, \sqrt[4]{15})$ we obtain

   $$y'|_{(1,\, \sqrt[4]{15})} = \frac{-1}{\sqrt[4]{15}^3}$$

   Thus the slope of the tangent at the point $(1, \sqrt[4]{15})$ is $\frac{-1}{\sqrt[4]{15}^3}$.

(b) $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the $(3,1)$;

BE CAREFUL: DIFFERENTIATE BOTH SIDE OF THE EQUALITY!!!!!!

We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get:

$$4(2x+2yy')(x^2+y^2) = 25(2x-2yy')$$

That is

$$y'(8y(x^2+y^2)-50y) = 50x = 8x(x^2+y^2)$$

Thus we obtain:

$$y' = \frac{8x(x^2+y^2)}{8y(x^2+y^2)-50y} = \frac{4x(x^2+y^2)}{y(4(x^2+y^2)-25)}$$

Then at the point $(3,1)$ we obtain

$$y'|_{(3,1)} = \frac{4\times 3(3^2+1^2)}{1(4(3^2+1^2)-25)} = \frac{4\times 3(10)}{15} = 8$$

Thus the slope of the tangent at the point $(3,1)$ is 8.